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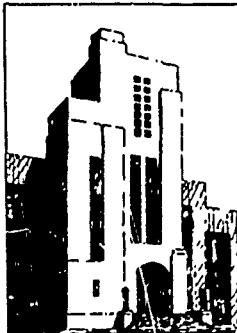
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CONTRA-ROTATING OPTIMUM PROPELLERS OPERATING
IN A RADIALLY NON-UNIFORM WAKE

by

H. W. Lerbs



May 1955

Report 941

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NOTATION

T	Total thrust	}
Q	Tangential force	
P	Total power	
P^*	Pitch	
z	Number of blades of one of the propellers	
$D = 2R$	Diameter	
$x = r/R$	Non-dimensional radial coordinate	
d	Distance of the two propellers	
δ	Contraction ratio	
v_s	Ship speed	
w	Effective wake factor	
w_o	Integrated effective wake factor	
$v = v_s(1-w)$	Local velocity of advance	
t	Thrust deduction factor	
t_o	Integrated thrust deduction factor	
w_a	Axial component of the induced velocity	
w_t	Tangential component of the induced velocity	
n	revolutions per second	
$\lambda_s = v_s/\pi n D$	Advance ratio	
ω	Angular velocity	
$c_T = \frac{T}{\frac{\rho R^2 \pi v_s^2}{2}}$	Thrust loading coefficient in non-viscous flow	

$c_P = \frac{P}{\frac{\rho R^2 \pi}{2} v_s^3}$	Power loading coefficient in non-viscous flow
c_L	Lift coefficient of any section
$\epsilon = c_D/c_L$	Drag-lift coefficient of any section
Γ	Bound circulation of one blade
$G = z\Gamma/\pi D v_s$	Non-dimensional bound circulation of one of the propellers
(PC)	Propulsive coefficient
f	Average factor
g	Distance factor
i	Induction factor
	A bar means an average value taken either circumferentially or radially

Indices:

1	Front propeller
2	Rear propeller
s	Self induction
i	Interference induction
h	Hub

INTRODUCTION

Known theories of a contra-rotating propeller are either restricted to uniform inflow (1)* or include arbitrary assumptions concerning both the applicability of the Goldstein function and the orientation of the resultant induced velocity relative to the free vortex sheets (2, 3). These assumptions are avoided in the following considerations which make use of the so-called induction factors of vortex sheets.

A criterion for optimum flow, expressed in terms of the direction of the free vortex sheets, is obtained from first order considerations. This criterion leads to a non-linear integral equation for the optimum circulation or, approximately, to a set of non-linear algebraic equations for the Fourier coefficients of the circulation. For uniform inflow, the free vortex sheets become of a true helical shape and the equations for the circulation reduce to a linear system.

A design method, which follows from the considerations, is outlined taking approximately into account the effects arising from the difference of the wake at the propeller discs and from the contraction of the wake between them.

Finally, the optimum circulation obtained by Theodorsen for uniform inflow by means of an electrical analogy (1) is compared with the result from the developed relations.

* The numbers in parentheses refer to the list of references on page 22

GENERAL CONSIDERATIONS

The flow at either propeller of a contra-rotating pair arises both from self induction and from interference induction (Figure 1).

At the front propeller, the interference causes an increase of the axial component of the relative velocity. There is no interference effect on the tangential component, as follows from Stokes' law.

At the rear propeller, both the axial and the tangential component of the relative velocity are increased by interference.

The following assumptions are made when establishing expressions for the flow components and for the generated forces:

(1) The shape of the vortex sheets is determined by the relative flow at the disc. This assumption requires both that the propellers are moderately loaded and that the wake does not appreciably change in the axial direction.

The first holds sufficiently if second and higher powers of the induced velocity may be neglected, the second if the frictional part of the wake is predominant.

(2) The self and mutual interference of the free vortex sheets may be neglected.

(3) The blades may be replaced by lifting lines and corrections for lifting surface effects may be introduced afterwards.

(4) The induced velocity components may be considered time averages, i.e., oscillations in both angle of attack and magnitude of velocity are not taken into account.

(5) The bound circulation may be represented by a Fourier sine series.

(6) The propellers have an equal number of blades. This latter assumption can be easily removed if deemed necessary.

COMPONENTS OF THE RELATIVE VELOCITY AND OF THE FORCES

From Figure 1, the components of the relative velocity at a blade element are for the front screw:

$$\text{axial} = v + (w_{as})_1 + (w_{ai})_1$$

$$\text{tangential} = \omega r - (w_{ts})_1$$

and for the rear screw:

$$\text{axial} = v + (w_{as})_2 + (w_{ai})_2$$

$$\text{tangential} = \omega r - [(w_{ts})_2 - (w_{ti})_2]$$

The interference velocities are considered circumferential averages, from assumption 4. These averages are obtained when multiplying the maximum value of the respective induced velocity component, which is induced at a bound vortex, by its average factor f and when introducing a second factor g to represent the change of the average in the vicinity of the disc. Correspondingly, the interference velocities may be expressed by the self-induced velocities as follows:

$$(w_{ai})_1 = (w_{as})_2 (f_a)_2 [1 - (g_a)_2] = (\bar{w}_{as})_2 [1 - (g_a)_2]$$

$$(w_{ti})_1 = 0$$

$$(w_{ai})_2 = (w_{as})_1 (f_a)_1 [1 + (g_a)_1] = (\bar{w}_{as})_1 [1 + (g_a)_1]$$

$$(w_{ti})_2 = 2 (w_{ts})_1 (f_t)_1 [1 + (g_t)_1] = 2 (\bar{w}_{ts})_1 [1 + (g_t)_1]$$

The factor 2 within the last relation arises from the discontinuity of the tangential component of the self induced velocity.

To approximately determine the factors g_a each propeller is replaced by a uniformly loaded sink disc and

$$(g_a)_1 \neq (g_a)_2$$

is assumed, i.e., symmetry of the gradient of the axially induced flow relative to the disc is assumed. From the known potential

of a sink disc it develops that

$$\left. \begin{aligned}
 g_a &= 1 - \cos \theta \left[\frac{1}{2} \frac{1}{b^2} - \frac{3}{8} \frac{1}{b^4} P_2 + \frac{5}{16} \frac{1}{b^6} P_4 - \frac{35}{128} \frac{1}{b^8} P_6 + \dots \right] \\
 [1] \quad &+ \sin \theta \left[\frac{1}{8} \frac{1}{b^4} P_2' - \frac{1}{16} \frac{1}{b^6} P_4' + \frac{5}{128} \frac{1}{b^8} P_6' + \dots \right]
 \end{aligned} \right\}$$

if $b > 1$ and

$$\begin{aligned}
 g_a &= 1 - \cos \theta \left[P_1 - b P_2 + \frac{1}{2} b^3 P_4 - \frac{3}{8} b^5 P_6 + \dots \right] \\
 &+ \sin \theta \left[P_1' - \frac{1}{2} b P_2' + \frac{1}{8} b^3 P_4' - \frac{1}{16} b^5 P_6' + \dots \right]
 \end{aligned}$$

if $b < 1$.

The functions P_n are the Legendre polynomials of the argument

$$\cos \theta = \frac{d/R}{b}$$

where

$$b = \sqrt{x^2 - (d/R)^2}$$

and where d represents the axial distance of the two propellers. The prime means the derivative of P relative to θ . The factor g_a is represented on Figure 2 as a function of x and d/R . The diagram indicates that the change of the axial component is rather great in the vicinity of the disc. This behaviour may necessitate replacing the approximations [1] by more rigorous expressions taking into account the non-uniform loading of the disc.

The factor g_t depends on contraction and is assumed to be zero for the present. This assumption will be corrected on page 12,
CORRECTIONS FROM THE WAKE DIFFERENCE AND FROM THE RAKE CONTRACTION.

The average factor f_t follows from Stokes' law. If (zT) represents the total bound circulation of one of the propellers,

this law yields the result;

$$2 r \pi (\bar{w}_{ts}) f_t = 2 r \pi (\bar{w}_{ts}) = z \Gamma$$

from which it follows that

$$[2] \quad f_t = \frac{(\bar{w}_{ts})}{w_{ts}} = \frac{z \Gamma}{4 r \pi (w_{ts})}$$

The average factor of the axial component, viz.,

$$f_a = \frac{(\bar{w}_{as})}{(w_{as})}$$

is not yet reliably known. Work for its determination has been initiated on a basis of the Biot-Savart integral for semi-infinite helical vortex lines. We put tentatively until accurate results are available:

$$[3] \quad f_a \doteq f_t$$

From these relations, it follows that the pitches of the free vortex sheets are slightly different for the front and rear propeller. To simplify, vortex sheets of equal shape are assumed the pitch of which may be taken as the mean value of $(\tan \phi_1)_1$ and $(\tan \phi_1)_2$.

A further simplification arises from the requirement that the torque of the front propeller equals that of the rear propeller. From the law of moment of momentum this condition is satisfied if

$$[4] \quad (z \Gamma)_1 = (z \Gamma)_2 = (z \Gamma)$$

The assumption involved, viz., that the propellers have equal diameters will be removed in CORRECTIONS FROM THE WAKE DIFFERENCE AND FROM THE RAKE CONTRACTION on page 12. The bound circulations of the two propellers being equal functions of x , the circulations of the free vortex sheets are also equal. Since the slight

difference between the pitch of these sheets is disregarded, the respective components of the induced velocities become equal. That is

$$(w_{as})_1 = (w_{as})_2 = w_a$$

$$(w_{ts})_1 = (w_{ts})_2 = w_t$$

With these simplifications, the following expressions for the components of the relative velocity are obtained -

$$[5] \left\{ \begin{array}{l} \text{for the front screw:} \\ \text{axial} = v + w_a [1 + f_a (1 - g_a)] \\ \text{tangential} = \omega r - w_t \\ \text{and for the rear screw:} \\ \text{axial} = v + w_a [1 + f_a (1 + g_a)] \\ \text{tangential} = \omega r - w_t (1 - 2 f_t) \end{array} \right.$$

Introducing a non-dimensional circulation, viz.,

$$[6] \quad G = \frac{z \Gamma}{\pi D v_s}$$

the thrust coefficient and the power input coefficient of the contra-rotating propeller are obtained from the law of Kutta-Joukowsky as follows:

$$[7] \quad c_T^* = 8 \int_{x_h}^1 G \left[\frac{x}{\lambda_s} - \frac{w_t}{v_s} (1 - f_t) \right] dx$$

$$[8] \quad c_P^* = \frac{8}{\lambda_s} \int_{x_h}^1 G \left[(1 - w) + \frac{w_a}{v_s} (1 + f_a) - x dx \right]$$

The loading coefficients so ascertained are slightly different from the given values c_t or c_p , respectively, from reasons which are discussed in CORRECTIONS FROM THE WAKE DIFFERENCE AND FROM THE RAKE CONTRACTION on page 12. Within these expressions, the average factors are determined from the relation

$$[9] \quad f_t = \frac{G}{2x \frac{w_t}{v_s}} \doteq f_a$$

Further, the self induced velocities w_a and w_t are related to G by the following expressions:

$$[10] \quad \left\{ \begin{array}{l} \frac{w_a}{v_s} = \frac{1}{2z} \int_{x_h}^1 \frac{dG}{dx_o} \frac{1}{x-x_o} i_a dx_o \doteq \frac{1}{z(1-x_h)} \sum_{m=1}^{\infty} m G_m^* h_m^a \\ \frac{w_t}{v_s} = \frac{1}{2z} \int_{x_h}^1 \frac{dG}{dx_o} \frac{1}{x-x_o} i_t dx_o \doteq \frac{1}{z(1-x_h)} \sum_{m=1}^{\infty} m G_m^* h_m^t \end{array} \right.$$

The functions i_a and i_t are the induction factors of the free vortex sheets as defined in (4). The functions h_m enter when the integral on the right hand side, which is an improper integral, is evaluated. These functions are related to the induction factors, the relations having been developed in (4). By the factors G_m^* , the coefficients of G are denoted when G is represented by a Fourier sine series within the interval $x \neq x_h$ and $x = 1$.

CRITERION FOR CONTRA-ROTATING PROPELLERS OF MINIMUM LOSS OF KINETIC ENERGY

The equations [7] to [10] together with an assumption on the thrust deduction factor suffice to treat the variational problem of a contra-rotating propeller operating in a given wake field, viz., to determine the coefficients of the Fourier series for $G(x)$ such that the useful work becomes a maximum value for a given power input. However, this problem of determining the optimum circulation function in a direct way is laborious for a finite

number of blades and is not solved here. Instead, a relation between β_1 and β is deduced for optimum flow from which the circulation function follows afterwards.

From considerations on moderately loaded single propellers, it is known that the form of the optimum relation between β_1 and β is independent of the number of blades. This enables us to consider propellers with infinitely many blades by which the deduction of the relation is greatly simplified. We apply the rule that the elementary propulsive coefficient which is related to an increment of circulation is independent of the radius for optimum flow, see (4). If the increments of the forces which arise from the increment of circulation, $\Delta(zT)$, are denoted by $\Delta(dt)$ and $\Delta(dQ)$, respectively, the rule reads:

$$\frac{\Delta(dt)}{\Delta(dQ)} \frac{v}{\omega r} \frac{1 - t(x)}{1 - w(x)} = k^2,$$

the constant k being independent of the radius x .

To establish $\Delta(dt)$ and $\Delta(dQ)$ we write from the law of Kutta-Joukowsky for infinitely many blades, i.e., for $f_a = f_t = 1$:

$$dT = dT_1 + dT_2 = 2(zT) \rho \omega r \quad dr$$

$$dQ = dQ_1 + dQ_2 = 2(zT) \rho (v + 2 w_a) \quad dr$$

It follows that

$$\Delta(dt) = 2 \rho \omega r \Delta(zT) \quad dr$$

and that

$$\Delta(dQ) = 2 \rho \left[(v + 2 w_a) + (zT)^2 \frac{\Delta w_a}{\Delta(zT)} \right] \Delta(zT) \quad dr$$

A relation between w_a and (zT) is obtained from the law of momentum. Neglecting terms of second power of w_a this relation reads:

$$w_a \approx \frac{r \omega}{v} \frac{(zT)}{4r \rho} (1 - t)$$

Then,

$$\Delta(dQ) \doteq 2\rho(v + 4w_a) \Delta(z\Gamma) dr$$

and the optimum condition becomes

$$\frac{\omega r}{v + 4w_a} = \frac{v}{\omega r} = \frac{k^2}{1 - t(x)}$$

Since the axial velocity in the slipstream far aft equals

$$v_\infty = v_s(1 - w) + 4w_a$$

the condition means that

$$v_\infty = \frac{v_s}{k^2} [1 - t(x)]$$

If the change of $t(x)$ over the radius is considered small so that $t(x) \approx t_0$, the condition requires that the axial velocity within the slipstream far aft is independent of the radius for infinitely many blades. These relations are approximations which include assumption 1, viz., that the change of the wake in the axial direction may be neglected.

To express the minimum condition by the pitch of the free vortex sheets we consider the average of the pitches of the two sheets. To the first order, this average for infinitely many blades amounts to

$$\tan \beta_1 \doteq \frac{v + 2w_a}{\omega r}$$

Since, to the first order

$$\frac{\omega r}{v + 4w_a} \doteq \left(\frac{\omega r}{v + 2w_a} \right)^2 = \frac{v}{\omega r}$$

the optimum condition may be written as

$$[11] \tan \beta_1 = \frac{\tan \beta}{k} \sqrt{\frac{1 - t(x)}{1 - w(x)}} = \frac{p(x)}{k}$$

That is, the mean pitch of the vortex sheets of a contra-rotating propeller follows to the first order the same law in the optimum case as established previously for a single wake adapted optimum propeller, see (5). However, the optimum circulation function of the contra-rotating propeller differs from that of a single wake adapted propeller as seen from the following section.

DETERMINATION OF THE CIRCULATION FUNCTION FOR OPTIMUM FLOW

Analogous to single propellers, the optimum relation as expressed by [11] is considered independent of the number of blades. Then, the optimum circulation function $G(x)$ is obtained for a finite number of blades when equating two expressions for $\tan \beta_i$ viz., the minimum condition [11] and the geometric relation;

$$\tan \beta_i \doteq \frac{v + w_a (1 + f_a) - \frac{v}{\omega r} w_t (1 - f_t)}{\omega r - 2 w_t (1 - f_t)}$$

This latter expression is a first order approximation for the mean value of $(\tan \beta_i)_1$ and $(\tan \beta_i)_2$ for a finite number of blades which follows from [5].

Introducing [9] for the average factors and equating, the relation

$$\begin{aligned} & \left[(1 - w) - \frac{x}{\lambda_s} \frac{p(x)}{k} \right] + \frac{G}{2x} \left[\frac{w_a}{w_t} + \frac{\lambda_s}{x} (1 - w) - \frac{2 p(x)}{k} \right] \\ &= \frac{w_t}{v_s} \left[\frac{\lambda_s}{x} (1 - w) - 2 \frac{p(x)}{k} \right] - \frac{w_a}{v_s} \end{aligned}$$

is obtained within which

$$p(x) = \tan \beta \sqrt{\frac{1 - t(x)}{1 - w(x)}} = \frac{\lambda_s}{x} \sqrt{(1-w)(1-t)}$$

is considered a known function of the radius.

Expressing w_a and w_t by the integral representation of [10] an integro-differential equation is deduced for G . An approximate solution of this equation becomes possible when utilizing the sums of [10] for w_a and w_t . This leads to the following system of algebraic equations for the Fourier coefficients G_m of G :

$$[12] \quad L_x = \sum_{m=1}^{\infty} G_m^* (M_{m,x} - N_{m,x})$$

where

$$L_x = (1 - x_h) \left[(1 - w) - \frac{x}{\lambda_s} \frac{p(x)}{k} \right]$$

$$M_{m,x} = \frac{m}{z} \left[h_t \left(\frac{\lambda_s}{x} \frac{L_x}{1-x_h} - \frac{p(x)}{k} \right) - h_m^a \right]$$

$$N_{m,x} = (1 - x_h) \frac{\sin m\varphi}{2x} \left[\frac{w_a}{w_t} + \left(\frac{\lambda_s}{x} \frac{L_x}{1-x_h} - \frac{p(x)}{k} \right) \right]$$

This relation is satisfied at m stations x in order to obtain m equations for m coefficients of the Fourier series:

$$[13] \quad G = \sum_{m=1}^{\infty} G_m^* \sin m\varphi$$

where

$$x = \frac{1}{2}(1 + x_h) - \frac{1}{2}(1 - x_h) \cos \varphi$$

On the right hand side of [12], the ratio w_a/w_t is not expressed by the sums of equation [10]. The reason is to avoid the numerical solution of non-linear equations for the coefficients G_m^* . This can be done when introducing for w_a/w_t as a first approximation the relation

$$\frac{w_a}{w_t} = \frac{1}{\tan \beta_1} \approx \frac{k}{p(x)},$$

i.e., assuming the condition of normality to be satisfied. This approximation may be corrected by successive solutions of the system [12].

It should be mentioned that the essential difference between a contra-rotating and a single wake-adapted optimum propeller, which is treated in (4), lies in the equations for G_m^* . These equations are non-linear for the former and linear for the latter.

CORRECTIONS FROM THE WAKE DIFFERENCE AND FROM THE RACE CONTRACTION

So far, the finite distance of the propellers has been taken into account by the distance factor g . The possible difference between the wakes at the positions of the propellers has been neglected and the radius of the streamlines has been assumed to be constant. Both the wake difference and the finite distance necessitate satisfying the equation of continuity within the space between the propellers. Corrections on the induced velocity components ensue and the diameters can no longer be assumed equal. With different diameters, slight changes of the bound circulation become necessary in order to maintain the condition of an equal torque for each propeller. We introduce the wake factor w_1 in the plane of the front propeller and the factor w_2 in that of the rear propeller so that

$$v_1 \approx v_s (1 - w_1)$$

$$v_2 \approx v_s (1 - w_2)$$

We assume that the wake does not change on the front propeller, i.e., that the wake factor w_1 is identical with the quantity w used in the foregoing considerations. Then, the results for both the self induced velocities from [10] and for G from [12] hold for the front propeller. since it will be seen that the slight change of the pitch of the free vortex sheets which ensues on the front propeller from the changes on the rear propeller may be neglected. Correspondingly, the index 1 is attached to these quantities and the non-dimensional circulation G_1 is defined by

$$[14] \quad G_1 = \frac{(zT)_1}{\pi D_1 v_s} \approx G$$

The equation of continuity requires the race to contract. A streamline which passes through equal radii on both propellers when ignoring contraction will now pass through the radius r_2 at the rear propeller. We write

$$[15] \quad r_2 \approx r_1(1 - \delta)$$

and consider both δ and $\frac{d\delta}{dr_1}$ small quantities. The so related

radii are denoted "corresponding" radii.

The question arises how both the self induced velocities and the interference velocities, as determined from [10] and [5] on a basis of an equal wake w_1 , change when the wake at the rear propeller is different from w_1 . The significant changes occur at the rear propeller. Relative to its self induction this problem is essentially that of a change of the induced velocity components when propellers of slightly different values of diameter, bound circulation and pitch of the free vortex sheets are considered. Let the bound circulation be $(z\Gamma)_1$ and $(z\Gamma)_2$ on corresponding radii of the front and rear propeller, respectively. We write

$$[16] \quad (z\Gamma)_2 = (z\Gamma)_1 (1 + \epsilon)$$

Both ϵ and $\frac{d\epsilon}{dr_1}$ are small quantities. From the integral representation of [10] it follows for the axially self induced component at the rear propeller, e.g., that

$$\begin{aligned} (w_{as})_2 &= \frac{1}{2\pi z} \int_{r_h}^{R_2} \frac{d}{dr_2} \left[z\Gamma_2(r_2) \right] (i_a)_2 \frac{d(r_o)_2}{r_2 - (r_o)_2} \\ &= \frac{1+\delta}{2\pi z} \int_{r_h}^{R_1} \left[\frac{d(z\Gamma)_1}{dr_1} (1+\epsilon) + \frac{d\epsilon}{dr_1} (z\Gamma)_1 \right] \left[(i_a)_1 + \frac{\delta(i_a)_1}{\delta\phi_1} \Delta\phi_1 + \frac{\delta(i_a)_1}{\delta(\frac{r_o}{r})} \right. \\ &\quad \left. \Delta\left(\frac{r_o}{r}\right) \right] \frac{d(r_o)_1}{r_1 - (r_o)_1} \end{aligned}$$

if $\delta(r_1)$ in [15] may be replaced by its mean value $\bar{\delta}$. Then $\Delta\left(\frac{r_o}{r}\right) = 0$. Further, the induction factors do not change

appreciably when the pitch of the free vortex sheets changes slightly so that the term containing $\frac{d(i_a)_1}{dr_1}$ may be neglected.

In addition, the product $\frac{d\zeta}{dr_1}(zT)_1$ is small of second order since both $\frac{d\zeta}{dr_1}$ and $(zT)_1$ are small, the latter because of moderate loading.

If ζ is replaced by its mean value $\bar{\zeta}$ the self induced velocities at the radius r_2 of the rear screw are approximately related to those at the corresponding radius of the front screw by

$$(w_{as})_2 \doteq (1 + \bar{\delta}) (1 + \bar{L}) (w_{as})_1$$

$$(w_{ts})_2 \doteq (1 + \bar{\delta}) (1 + \bar{L}) (w_{ts})_1$$

From these equations, the afore mentioned slight change of the pitch of the free vortex sheets on the front propeller follows. The reason for the change is that the interference from the rear propeller is different when the self induced velocities of the latter are different. However, this effect may be neglected because of the negligible variation of the induction factors which is involved so that the result from [10] obtained with $w_1 = w$ is sufficiently accurate for the self induction on the front propeller.

To determine the interference velocities from the expressions for the self induced velocities the changes of the factors f and g must be known. The change of the average factor f on the rear propeller arises from the changes of the circulation, of the radius and of the self induced velocity. It follows from [2], however, that the resultant change of f is small of second order so that

$$f_t(r_2) \doteq f_t(r_1)$$

$$f_a(r_2) \doteq f_a(r_1)$$

The change of the distance factors g_a and g_t arises from the contraction of the slipstream. For the factor g_a , the effect is included in equations [1] since these equations represent the rigorous solution for the inflow to the sink disc. For the front screw, g_a is determined at the radius r_1 and for the rear screw at the radius r_2 . However, the accuracy of these equations is less for

the flow behind a propeller than in front because of the assumed symmetry of the gradient of the axially induced flow. Because of this, the approximation

$$g_a(r_2) \approx g_a(r_1)$$

is considered to be in accord with the applicability of equations [1].

The effect of contraction on the factor g_a is established from the law that the circulation remains unaltered on a contracting streamline.

The components of the interference induction are then related to the self induction as follows:

$$(w_{ai})_1 = (\bar{w}_{as})_2 (1 - g_a)$$

$$(w_{ti})_1 = 0$$

$$(w_{ai})_2 = (\bar{w}_{as})_1 (1 + g_a)$$

$$(w_{ti})_2 = 2(\bar{w}_{ts})_1 (1 + \delta)$$

From these expressions for the self and interference induction the components of the relative velocity follow at corresponding radii. One obtains

[17] {

for the front screw (radius r_1)	$\text{axial} = v_s + (w_{as})_1 + (w_{ai})_1$ $= v_s (1 - w_1) + (w_{as})_1 \left[1 + f_a (1 + \bar{\delta}) (1 + \bar{\delta}) (1 - g_a) \right]$ $= v_s A_1$
tangential = $\omega r_1 - (w_{ts})_1$	
for the rear screw (radius r_2 corresponding to r_1)	$\text{axial} = v_s + (w_{as})_2 + (w_{ai})_2$

$$= v_s (1 - w_2) + (w_{as})_1 \left[(1 + \bar{\zeta}) (1 + \bar{\delta}) + f_a (1 + g_a) \right]$$

$$= v_s A_2$$

$$\text{tangential} = \omega r_2 - (w_{ts})_2 + (w_{ti})_2$$

$$= \omega r_1 (1 - \delta) - (w_{ts})_1 \left[(1 + \bar{\zeta}) (1 + \bar{\delta}) - 2 f_t (1 + \bar{\delta}) \right]$$

where $(w_{as})_1$ and $(w_{ts})_1$ are the result from [10] with $w = w_1$ and $r = r_1$.

The contraction $\delta(r_1)$ is ascertained from the equation of continuity which reads for an annular element as follows:

$$\left[v_1 + (\bar{w}_{as})_1 + (w_{ai})_1 \right] r_1 dr_1 = \left[v_2 + (\bar{w}_{as})_2 + (w_{ai})_2 \right] r_2 dr_2$$

Introducing the expression [15] and the relations for (\bar{w}_{as}) and (w_{ai}) a first order linear differential equation for δ is obtained when neglecting terms which are small of second order. Determining the constant of integration such that $\delta = 0$ for $x_1 = (x_h)_1$ the solution is represented by the following integral:

$$[18] \delta = \frac{1}{x_1^2} \int_{(x_h)_1}^{x_1} \frac{(w_1 - w_2) + \frac{(w_{as})_1}{v_s} f_a g_a [2 + (\bar{\delta} + \bar{\zeta})]}{(1 - w_2) + \frac{(w_{as})_1}{v_s} f_a [2 + (\bar{\delta} + \bar{\zeta} + g_a)]} dx_1$$

It remains to establish a relation for the function $\zeta(r_1)$ which follows from the condition of an equal torque for each propeller. From the law of Kutta-Joukowsky the balance requires that

$$[19] \zeta = \frac{A_1}{A_2} \frac{1}{1 - 2\delta - x_1 \frac{d\delta}{dx_1}} = 1$$

This equation is in accord with the law of moment of momentum which necessitates that $\zeta = 0$ if the propellers are very close together, i.e., if both $\delta = 0$ and $g_a = 0$.

The functions δ and ζ are interrelated within [18] and [19]. These equations may be solved in successive steps assuming in the first step δ , ζ and ζ to be zero. The solutions for $\delta(x_1)$ and $\zeta(x_1)$ so obtained lead to first approximations for δ and ζ with which improved values of $\delta(x_1)$ and $\zeta(x_1)$ follow.

With δ being known the change of the diameter may be determined which is required by the equation of continuity. This equation is satisfied if

$$[20] \quad D_2 / D_1 = (1 - \delta)_{x_1} = 1$$

Finally, the alteration of the total thrust may be ascertained which ensues from the changes under consideration. Between the thrust coefficient c_T^* as follows from [7] and the coefficient c_T obtained from the law of Kutta-Joukowsky when introducing equations [15] to [17] the approximation

$$[21] \quad c_T \approx c_T^* \left[1 - \left(\frac{\delta}{2} - \frac{\zeta}{2} \right) \right]$$

holds.

DESIGN PROCEDURE

From the developed relations, a design may proceed in the following way considering the total thrust a given quantity which is denoted c_T . The first step is to solve the algebraic equations [12] for the coefficients G_m^* assuming several values of the constant k and to determine k such that the thrust coefficient c_T^* is obtained from [7]. The quantity c_T^* in [7] is related to the required quantity c_T by [21]. Since both δ and ζ are unknown when beginning the calculation an estimate is necessary to determine c_T^* . A slight increase of c_T will usually be sufficient to satisfy [21].

The constant k is approximately ascertained by the following relation, see (5):

$$k \approx (PC) \sqrt{\frac{1-w_0}{1-t_0}}$$

so that

$$\tan \beta_i = \frac{p(x_1)}{k} = \frac{\lambda_s}{x_1} \frac{1-t_0}{(PC)} \sqrt{\frac{1-w(x_1)}{1-w_0}}$$

These relations make it possible to approximately determine k and $\tan \beta_i$ from estimated values of both the propulsion coefficient and t_0 . The assumption $t(x) \approx t_0$ made in the expression for $\tan \beta_i$ can be improved utilizing results of (6). However, the influence of this assumption on the coefficients G_m^* is small.

With this first approximation for $\tan \beta_i$ the functions i and h are established in a way which is analogous to that shown in (4) for a single propeller. Next, the equations [12] are solved and the circulation function $G(x)$, which is identical with $G_1(x_1)$, is calculated from [13]. The components of the self induced velocity follow from the sums of [10]. The average factors are then known from [9] and the thrust loading coefficient from [7]. If the difference between the required and the calculated value for c_T^* is appreciable the calculations are repeated using a slightly different value for k .

After sufficient agreement between the two values for c_T^* is obtained both the contraction δ and the function ζ are found from [18] and [19]. The diameter D_2 follows from [20], D_1 being assumed from the beginning. The pitches and the products $(c_T^* l)$ are then ascertained from the following relations which hold at the radii $x_1 = r_1/R_1$ and $x_2 = r_2/R_2 = x_1 \frac{1-\delta}{(1-\delta)_{x=1}}$ respectively:

$$\left\{ \begin{array}{l} (\tan \beta_i)_1 = \frac{v_1 + (w_{as})_1 + (w_{ai})_1}{\omega r_1 - (w_{ts})_1} = \frac{A_1}{\frac{x_1 - (w_{ts})_1}{\lambda_s - \frac{v_s}{v_s}}} \\ (\tan \beta_i)_2 = \frac{v_2 + (w_{as})_2 + (w_{ai})_2}{\omega r_2 - (w_{ts})_2 + (w_{ti})_2} \end{array} \right.$$

$$= \frac{A_2}{\frac{x_2 (1 - \delta)}{x_1 z_1} - \frac{(w_{ts})_1}{v_s} \left[(1 + \zeta) (1 + \delta) - 2 f_t (1 + \delta) \right]}$$

where

$$\lambda_s = \frac{v_s}{R_1 \omega}$$

$$\left\{ \begin{array}{l} \frac{P_1^*}{D_1} = \pi x_1 \tan (\beta_{11} + \alpha_1) \\ \frac{P_2^*}{D_2} = \pi x_2 \tan (\beta_{12} + \alpha_2) \end{array} \right.$$

$$\left\{ \begin{array}{l} \frac{(c_L \ell)_1}{D_1} = \frac{2\pi}{z} \frac{G_1}{A_1} (\sin \beta_{11})_1 \\ \frac{(c_L \ell)_2}{D_2} = \frac{2\pi}{z} \frac{G_1}{A_2} (1 + \zeta) (\sin \beta_{12})_2 \end{array} \right.$$

The design may be checked by means of the following expressions which hold for viscous flow:

$$(c_T)_{visc} = \frac{T_{visc}}{\frac{1}{2} v_s^2 R_1^2 \pi} = \frac{2z}{\pi} \left[B_1 + (1 - \delta)_{x_1=1} B_2 \right]$$

where $B = \int_{(x_h)}^1 \frac{(c_L \ell)}{D_1} \left(\frac{V}{v_s} \right)^2 (\cos \beta_{11}) (1 - \epsilon \tan \beta_{11}) (dx)$

and

$$(V/v_s) = (A / \sin \beta_{11})$$

using indices 1 or 2 on the terms in brackets to obtain B_1 or B_2 respectively.

$$(c_p)_{visc} = \frac{P_{visc}}{\frac{1}{2} v_s^3 R_1^2 \pi} = \frac{2z}{\pi} \frac{1}{\lambda_s} \left[C_1 + (1 - 2\delta)_{x=1} C_2 \right]$$

$$C = \begin{cases} \frac{1}{D_1} \left(\frac{c_L l}{v_s} \right)^2 \frac{(\sin \beta_1) (1 + \frac{E}{\tan \beta_1})}{\tan \beta_1} & (x \neq x_h) \\ (x_h) & \end{cases}$$

The former equation may be used to check whether the numerical value for the ratio $c_T/(c_T)_{visc}$ is reasonably assumed when starting the computations. The latter relation gives an estimate of the power input and, considering the integrals separately, can also be used to check whether the balance of torque is maintained in viscous flow. As in the design of single propellers, corrections on the sections for lifting surface effects (additional camber and additional angle of attack) and for viscous flow (additional angle of attack) are introduced when determining pitch and camber. Also, the effect of the hub on the induced velocity may be estimated as outlined in (4).

OPTIMUM CIRCULATION FUNCTION FOR UNIFORM INFLOW AND COMPARISON WITH THEODORSEN'S RESULTS

For uniform inflow, i.e., for

$$w = 0, t = 0, p(x) = \frac{\lambda}{x}$$

the minimum condition [11] reduces to

$$\tan \beta_1 = \frac{1}{k} \frac{\lambda}{x}$$

That is, the mean pitch of the vortex sheets forms a true helical surface. Consequently, the condition of normality is satisfied so

that the relation

$$\frac{w_a}{w_t} = k \frac{x}{\lambda}$$

holds in this case. Then, the equations [12] for the optimum circulation are simplified to the following linear system assuming $x_h = 0$:

$$L = \sum_{m=1}^{\infty} G_m^* (M_{m,x} + N_{m,x})$$

where

$$L = 1 - \frac{\lambda}{\lambda_i}$$

$$M_{m,x} = \frac{m}{z} \left[\frac{\lambda}{x} \left(2 - \frac{\lambda}{\lambda_i} \right) h_m^t + \frac{\lambda}{\lambda_i} h_m^a \right]$$

$$N_{m,x} = \frac{\sin(m\varphi)}{2x} \left[\left(\frac{\lambda}{\lambda_i} \right)^2 \frac{x}{\lambda} - \left(2 - \frac{\lambda}{\lambda_i} \right) \frac{\lambda}{x} \right]$$

This system is solved numerically for the arbitrary values $z = 4$, $\lambda = 0.80$, $\lambda_i = 0.94$. The result obtained is compared with that determined by Theodorsen for hubless propellers by means of an electrical analogy which is based on the fact that a velocity potential may be reproduced by an electrical potential provided that the boundary conditions are identical (1).

The following solution of the set of equations is obtained:

$$G_1^* = .11907 \quad G_4^* = .00024$$

$$G_2^* = .00844 \quad G_5^* = .00005$$

$$G_3^* = .00417$$

The ensuing circulation distribution G is represented on Figure 3. As compared to the circulation following from Theodorsen's work, differences arise near $x=0$ where G goes to zero

whereas Theodorsen's function remains finite. This difference arises from the representation of G by a Fourier sine series which is not suited to represent the optimum circulation of a hubless contra-rotating propeller close to the axis. However, for the essential part of the blade, between x about 0.3 and 1, the agreement is considered satisfactory. This result indicates that the optimum circulation of a contra-rotating propeller with a finite hub is sufficiently represented by the series chosen for G also close to the hub.

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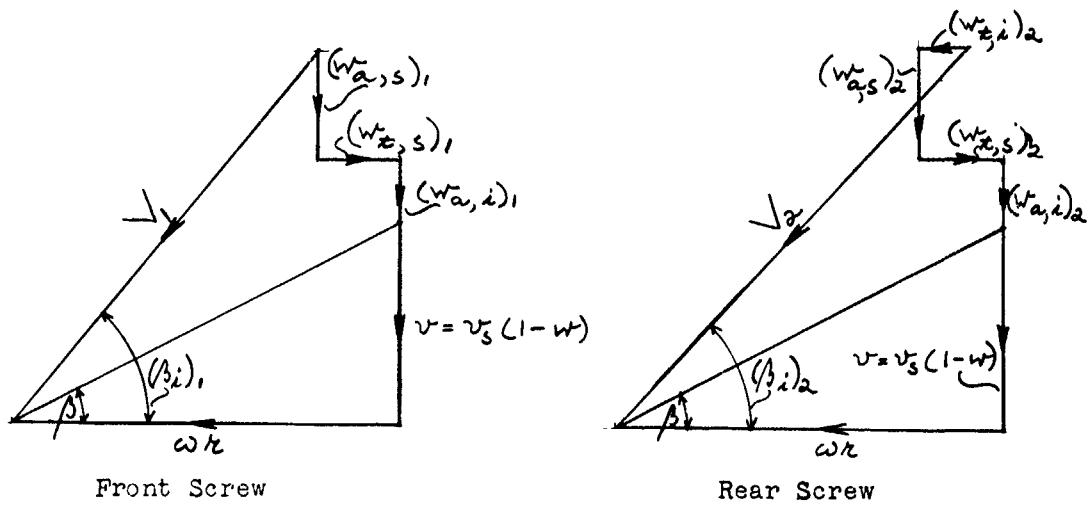


Figure 1

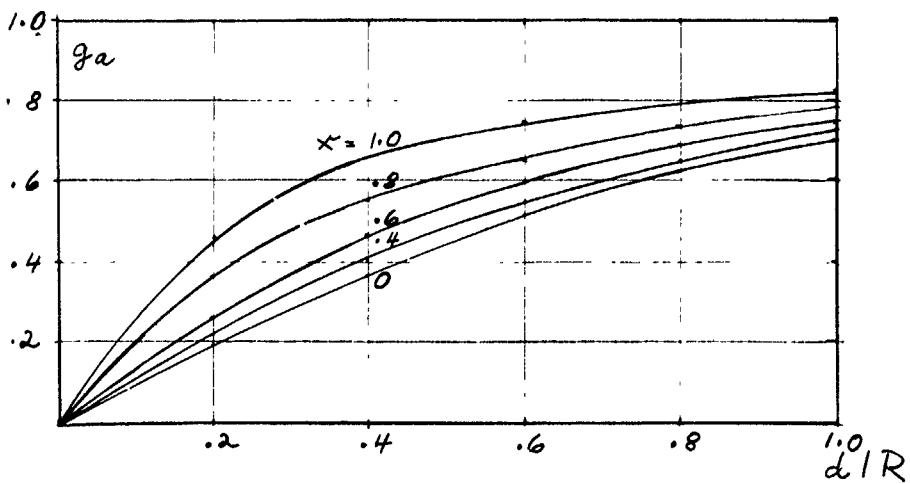


Figure 2

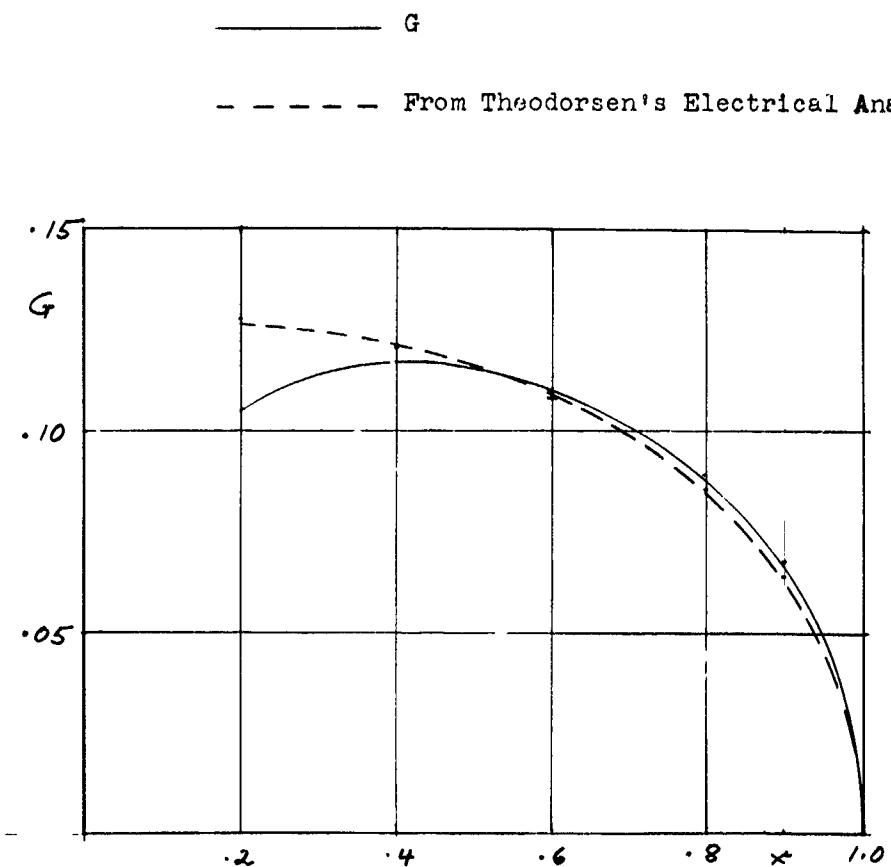


Figure 3

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